System of implicit equations, IS-LM Model

Given the IS-LM model:

$$
\begin{aligned}
Y &= C(Y) + I(r) + g \\
M_p &= L(Y, r) \\
Y_d &= Y - T(Y, \theta)
\end{aligned}
$$

The endogenous variables are income Y, interest rate r and disposable income Y_d; the exogenous variables are M_p and public spending g. θ is a parameter. The model is assumed to present continuous partial derivatives with the following signs:

$$
0 < \frac{\partial C}{\partial Y_d} < 1, \quad \frac{\partial I}{\partial r} < 0, \quad \frac{\partial L}{\partial Y} > 0, \quad \frac{\partial L}{\partial r} < 0, \quad 0 < \frac{\partial T}{\partial Y} < 1, \quad \frac{\partial T}{\partial \theta} > 0
$$

1. Combine the third and first equations to obtain a model with two equations.
2. Verify that the model satisfies the condition of the implicit function theorem.
3. Determine the effect of an increase in public spending on the endogenous variables Y and r.

Solution

1. Replace Y_d in the first equation:

$$Y = C(Y - T(Y, \theta)) + I(r) + g$$

   Isolate the two equations:

$$C(Y - T(Y, \theta)) + I(r) + g - Y = 0$$

$$M_p - L(Y, r) = 0$$

2. Differentiate both equations with respect to variables: Y, r, M_p, and g.

$$I'rdr + dg + (C'Y(1 - T'Y) - 1)dY + 0dM_p = 0$$

$$-L'rdr + 0dg - L'YdY + dM_p = 0$$

   Isolate the exogenous from the endogenous variables:

$$I'rdr + (C'Y(1 - T'Y) - 1)dY = -dg$$

$$L'rdr + L'YdY = dM_p$$

   We set $dM_p = 0$ to see the effect of an increase in g on Y and r.

$$I'rdr + (C'Y(1 - T'Y) - 1)dY = -dg$$

$$L'rdr + L'YdY = 0$$

   Write the system in matrix form:

$$\begin{bmatrix} I'r & (C'Y(1 - T'Y) - 1) \\ L'r & L'Y \end{bmatrix} \begin{bmatrix} \frac{\partial r}{\partial g} \\ \frac{\partial Y}{\partial g} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

   Obtain the determinant of the Jacobian:

$$|J| = I'rL'Y - L'r(C'Y(1 - T'Y) - 1)$$

   Determine the sign of the Jacobian:

$$|J| = \overbrace{I'r}^{-}\,\overbrace{L'Y}^{+} - \overbrace{L'r}^{-}\,(\overbrace{C'Y}^{+}(1 - \overbrace{T'Y}^{+}) - 1)$$

   Also knowing that some terms are between 0 and 1.

$$|J| = \overbrace{I'rL'Y}^{-}\,\overbrace{-L'r(C'Y(1 - T'Y) - 1)}^{-} < 0$$

   Since the Jacobian is non-zero, the implicit function theorem holds, and thus we can find the partial derivatives.

3. Now obtain the derivatives, by Cramer's rule:

$$\frac{\partial Y}{\partial g} = \frac{\begin{vmatrix} I'r & -1 \\ L'r & 0 \end{vmatrix}}{|J|} = \frac{\overbrace{L'r}^{-}}{\underbrace{|J|}_{-}} > 0$$

$$\frac{\partial r}{\partial g} = \frac{\begin{vmatrix} -1 & (C'Y(1-T'Y)-1) \\ 0 & L'Y \end{vmatrix}}{|J|} = \frac{\overbrace{-L'Y}^{+}}{\underbrace{|J|}_{-}} > 0$$

Therefore, an increase in spending increases both the interest rate and income.